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Prolongation Structures and Conservation Laws of a Nonlinear Evolution Equation

Mohammed Banaga Al Basheir

Umm Al Quwain University

Umm Al Quwain University, Umm Al Quwain, United Arab Emirates

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Abstract: **We present a geometric and algebraic investigation of a nonlinear evolution equation using the framework of exterior differential systems and Wahlquist- Estabrook prolongation method. The equation is formulated as a closed differential ideal generated by differential 2-forms, and its compatibility conditions are analyzed on an extended manifold. This analysis reveals strong algebraic constraints that force the associated prolongation generators to vanish, thereby excluding the existence of a nontrivial classical prolongation structure. In addition, a search for local conservation currents in the form of closed 1-forms shows that only trivial conserved quantities arise. These results indicate that the equation does not admit structural features typically associated with complete integrability within the present framework.**

Keyword: **Nonlinear Evolution Equation; Prolongation Structure; Integrability; Exterior Differential System; Conservation Laws.**

I. INTRODUCTION

Nonlinear evolution equations play a central role in mathematical physics and differential geometry, where they model a wide variety of diffusion, transport, and reaction phenomena. Of particular interest are equations that possess rich geometric structures, such as prolongation representations and conservation laws, since these features often indicate complete integrability and deeper analytic properties of such systems [1-3]. The geometric interpretation of integrable systems has provided powerful insights into their structure. Sasaki showed that a broad class of soliton equations admits a description in terms of pseudospherical surfaces of constant negative Gaussian curvature [4]. Similarly, Chern demonstrated that the sinh-Gordon equation has a natural geometric interpretation in the theory of surfaces of constant curvature [15]. Integrable models such as the sine-Gordon and sinh-Gordon equations have become fundamental examples in the study of soliton theory and geometric methods for nonlinear partial differential equations [4-6,15].

Prolongation methods and conservation laws provide powerful tools for investigating the integrability properties of nonlinear systems. These approaches have been successfully applied to a variety of evolution equations, revealing deep connections between Lie algebras, prolongation structures, and conservation laws [7, 9, 10, 16-18]. In particular, exterior differential systems provide a natural geometric framework for the study of such structures and their compatibility conditions. At the same time, symmetry methods and conservation laws have become central techniques for characterizing qualitative and structural properties of nonlinear differential equations [11-14].

In this paper, we investigate the nonlinear evolution equation

$$u_t - u_{xx} + \sinh(u) = 0, \tag{1}$$

using the framework of exterior differential systems and the Wahlquist- Estabrook prolongation method. The equation is formulated as a closed differential ideal generated by differential forms, and the associated compatibility conditions are analyzed on an extended manifold. We show that the resulting algebraic constraints force the associated prolongation generators to vanish and lead only to trivial local conservation laws within the chosen ansatz. Consequently, the equation does not admit a nontrivial classical prolongation structure within the present framework, providing a clear characterization of its integrability limitations within the present framework.

II. THE EXTERIOR DIFFERENTIAL SYSTEM

We begin by reformulating the equation (1) within the framework of exterior differential systems, which provide a geometric method for representing nonlinear partial differential equations through differential forms and differential ideals and have been successfully applied to a variety of nonlinear evolution equations [7-10]. We introduce an auxiliary variable z representing the first spatial derivative $z = u_x$.

This allows us to express the equation (1)

$$u_x = z, \qquad z_x - u_t = \sinh(u) \qquad 2.$$

We establish an exterior differential system on a base manifold $M = \mathbb{R}^4$ with local coordinates $\{x, t, u, z\}$ that support differential forms. Such constructions are standard in the geometric theory of exterior differential systems and provide a convenient framework for investigating integrability properties of nonlinear equations [8, 10]

We construct the differential ideal I generated by the following exterior 2-forms

$$\alpha_1 = du \wedge dt - zdx \wedge dt$$

$$\alpha_2 = dz \wedge dt + du \wedge dx - \sinh(u)\, dx \wedge dt \qquad 3.$$

The differential forms (3) are chosen so that their integral manifolds correspond to solutions of the original evolution equation, following the standard exterior differential systems approach [7-10].

The exterior derivatives of the α_j can be written as

$$d\alpha_1 = -dz \wedge dx \wedge dt = \alpha_2 \wedge dx$$

$$d\alpha_2 = -\cosh(u)\, du \wedge dx \wedge dt = \cosh(u)\, du \wedge dt \wedge dx = \cosh(u)\, \alpha_1 \wedge dx \qquad 4,$$

so that the ideal generated by $\{\alpha_1, \alpha_2\}$ is closed under exterior differentiation, that is

$$d\alpha_i = \sum_{j=1}^{2} h_j \wedge \alpha_j \qquad 5,$$

where $h_j = h_j(x, t)$ are certain differential 1-forms. This shows that the differential ideal generated by $\{\alpha_1, \alpha_2\}$ is closed under exterior differentiation, which is the fundamental compatibility condition in the theory of exterior differential systems [8, 10]. On a transversal integral manifold, it follows that the differential system (3) can be sectioned to give

$$\alpha_1|_s = (u_x - z)\, dx \wedge dt = 0$$

$$\alpha_2|_s = (u_t - z_x + \sinh(u)\,)\, dx \wedge dt = 0 \qquad 6,$$

The transversal integral manifolds correspond to the equations (2).

These exterior differential formulations have been employed in the study of generalized Korteweg-de Vries equations and other nonlinear evolution equations, where they provide a natural starting point for prolongation analysis carried out in the next section [7-9].

III. PROLONGATION STRUCTURES

We now construct a Wahlquist-Estabrook type prolongation structure associated with the differential ideal (3). The prolongation method provides a geometric procedure for extending a differential system through pseudopotential variables and has proved to be an effective tool in the analysis of nonlinear evolution equations and their integrability properties [1, 2].

Following the standard prolongation approach, we introduce the pseudopotential 1-form

$$\omega = dy - F(x,t,u,z,y)dx - G(x,t,u,z,y)dt, \qquad \omega = dy - \theta, \qquad \theta = A(x,t)dx + B(x,t)dt \qquad 7,$$

where y is pseudopotential. This construction follows the classical Wahlquist-Estabrook prolongation approach, in which compatibility conditions on an extended manifold determine the admissible prolongation algebra associated with the differential system [1, 2].

the exterior derivative $d\omega$ lies in the ideal generated by the forms α_j and ω, that is

$$d\omega = \sum_{j=1}^{2} \lambda_j \alpha_j + \theta \wedge \omega\,. \qquad 8.$$

The requirement that $d\omega$ belongs to the ideal generated by $\{\alpha_1, \alpha_2, \omega\}$ constitutes the fundamental prolongation condition and guarantees the consistency of extended differential system [1, 2, 9].

Expanding the right-hand side of (8) yields

$$\sum_{j=1}^{2} \lambda_j \alpha_j + \theta \wedge \omega = \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + (Adx + Bdt) \wedge (dy - Fdx - Gdt)$$

$$= \lambda_1(du \wedge dt - z\, dx \wedge dt) + \lambda_2(dz \wedge dt + du \wedge dx - \sinh(u)\, dx \wedge dt) + A\, dx \wedge dy + B\, dt \wedge dy - AG\, dx \wedge dt + BF\, dx \wedge dt, \qquad 9,$$

and the left-hand side of (8)

$$d\omega^i = d\omega = d[dy - F(u,x,t,z,\mathrm{y})dx - G(u,x,t,z,\mathrm{y})dt] = -dF \wedge dx - dG \wedge dt$$

$$= F_u dx \wedge du - F_t dt \wedge dx - F_z dz \wedge dx - F_{\mathrm{y}} d\mathrm{y} \wedge dx - G_u du \wedge dt - G_x dx \wedge dt - G_z dz \wedge dt - G_{\mathrm{y}} d\mathrm{y} \wedge dt \quad 10.$$

Comparing coefficients of the independent 2-forms on both sides yields the system

$$F_u = -\lambda_2, \qquad F_z = 0, \qquad F_{\mathrm{y}} = A, \qquad G_u = \lambda_1, \qquad G_z = -\lambda_2, \qquad G_{\mathrm{y}} = B,$$

$$-\lambda_1 z - \lambda_2 \sinh(u) - AG + BF = F_t - G_x$$

$$zG_u + F_u \sinh(u) - \left[F_{\mathrm{y}} G - G_{\mathrm{y}} F\right] = F_t - G_x$$

$$zG_u + F_u \sinh(u) - [F, G] = F_t - G_x\,, \qquad 11,$$

where $F_y G - G_y F = [F, G]$. To obtain an explicit prolongation, we restrict to the class where F and G are independent of (x, t). In this case, the system (11) simplifies to

$$F = F(u, y), \qquad G = G(u, z, y)$$

$$G_z = F_u, \qquad [F, G] = zG_u + F_u \sinh(u) \qquad 12,$$

Such reductions are commonly employed in prolongation analyses in order to isolate the essential algebraic structure generated by pseudopotential variables [9, 16].

Integrating the first equation (12) with respect to z, yields

$$G(u, z, y) = zF_u(u, y) + H(u, y) \qquad 13,$$

where H is an arbitrary function. Differentiating (13) with respect to u, we get

$$G_u = zF_{uu} + H_u \qquad 14.$$

Substituting (13) and (14) into the second equation in (12), we obtain

$$[F, zF_u + H] = z(zF_{uu} + H_u) + F_u \sinh(u) \qquad 15$$

$$z[F, F_u] + [F, H] \ = z^2 F_{uu} + zH_u + F_u \sinh(u) \qquad 16.$$

Since F and H are independent of z, the coefficient of z^2 must vanish, hence $F_{uu} = 0$

This can be solved for F to give

$$F(u, y) = uX_1(y) + X_2(y) \qquad 17$$

where the $X_i(y)$ are vertical vector fields. Consequently (16) simplifies to

$$z[F, F_u] + [F, H] = zH_u + F_u \sinh(u) \qquad 18.$$

The coefficient z implies that

$$[F, F_u] = H_u \qquad 19$$

which using (17) immediately establishes basic commutators relations among the vectors field

$$H_u = [uX_1 + X_2, X_1] = u[X_1, X_1] + [X_2, X_1]$$

$$H_u = [X_2, X_1] \qquad 20$$

Integrating (20) with respect to u yields

$$H(u, y) = -u[X_1, X_2] + X_3(y) \qquad 21$$

The appearance of the vertical vector fields X_i reflects the Lie-algebraic nature of the prolongation construction, where compatibility conditions generate commutation relations among prolongation generators [1, 2,16].

The coefficient z^0 implies the condition

$$[F, H] = F_u \sinh(u) \qquad 22$$

Substituting (17) and (21) into (19) and using linearity to expand out, we get

$$[uX_1 + X_2, -u[X_1, X_2] + X_3] = X_1 \sinh(u) \qquad 23$$

$$-u^2\big[X_1, [X_1, X_2]\big] + u\big([X_1, X_3] - \big[X_2, [X_1, X_2]\big]\big) + [X_2, X_3] = X_1 \sinh(u)$$

the left-hand side is polynomial of degree 2, while the right-hand side contains the full Taylor expansion of $\sinh(u)$. Therefore, the identity can hold only if

$$X_1 = 0, \qquad F_u = 0, \qquad H_u = 0, \qquad [X_2, X_3] = 0$$

and the prolongation structure is trivial. This proves that the equation does not admit a nontrivial classical prolongation structure.

This conclusion differs significantly from the situation encountered in classical integrable equations, where rich finite and infinite dimensional prolongation algebras arise naturally form the corresponding compatibility conditions [17].

IV. CONSERVATION LAWS

Conservation laws constitute one of the most important structural indicators of integrability in nonlinear partial differential equations. Within the framework of differential forms, they are often characterized by closed differential forms whose existence reflects invariant geometric properties of the underlying system [11-14].

We consider a differential 1-form γ satisfying the closure condition

$$d\gamma = 0.$$

This is the integrability condition for the existence of a potential ω such that $\gamma = d\omega$, which implies

$$d\gamma = d^2\omega = 0.$$

We first investigate conservation laws arising directly from the differential ideal by considering linear combinations of the basic 2-forms [10,13]

$$\varphi = \sum_{j=1}^{2} h_j \alpha_j \qquad 24,$$

where $h_i = h_i(u, z)$, and (24) satisfies the exactness condition $d\varphi = 0$, yields a system of equations for h_1 and h_2. Solving these relations shows that

$$h_1 = h_2 = 0 \qquad 25.$$

Therefore, there are no nontrivial conservation laws of the form $\varphi = h_1\alpha_1 + h_2\alpha_2$.

This indicates that the differential ideal itself does not generate nontrivial local conservation laws under the present ansatz.

Next, we search for local conservation laws in the standard density-flux representation [11,14]

$$\gamma = f(u,z)dx + g(u,z)dt, \qquad z = u_x \qquad 26,$$

where f represents the conserved density and g the flux. This relation is characteristic of local conservation law calculations and consistent with the direct-construction method commonly employed in the literature [13, 14].

The closure condition $d\gamma = 0$, together with

$$u_t = z_x - \sinh(u) \qquad 27,$$

gives

$$(g_z - f_u)\, z_x - f_z\, z_t + f_u \sinh u + z g_u = 0 \qquad 28.$$

Since this identity must hold on solutions, we obtain

$$g_z = f_u, \qquad f_z = 0 \qquad 29.$$

Hence $f = f(u)$, and integrating the first relation (29) with respect to z, we get

$$g(u,z) = z\, f_u(u) + h(u) \qquad 30,$$

where $h(u)$ is an arbitrary function of u.

Substituting (30) into (28), we get

$$f_u \sinh u + z\big(z\, f_{uu} + h'(u)\big) = 0 \qquad 31.$$

Since the identity must hold for all z, we conclude that

$$f_{uu} = 0, \qquad h'(u) = 0, \qquad f_u \sinh u = 0 \Longrightarrow \sinh u \neq 0, \Longrightarrow f_u = 0 \qquad 32.$$

Consequently, the density f and the flux g reduce to arbitrary constants, yielding the exact form $\gamma = c_1 dx + c_2 dt$ is trivial.

This confirms the algebraic rigidity of the system and shows that, within the chosen ansatz, it admits no nontrivial local conservation laws depending on the dynamic variable. This result is limited to local currents of the form $\gamma = f(u,z)dx + g(u,z)dt$, while the existence of nonlocal conservation laws remains open. It is worth noting that nonlocal conservation laws [18] have been successfully derived for the sinh-Gordon equation through Backlund transformations and nonlocal symmetry techniques, leading to infinite hierarchies of conserved quantities.

V. CONCLUSION

The nonlinear evolution equation was formulated as a closed exterior differential system and examined through the Estabrook-Wahlquist prolongation method. The compatibility conditions were found to eliminate all nontrivial prolongation generators, while the analysis of closed 1-forms produced only trivial local conservation laws. Consequently, no nontrivial classical prolongation structure and no nontrivial local conservation law were detected within the present framework. which clarifies the integrability limitation of the model.

REFERENCES

[1] Wahlquist HD, Estabrook FB. Prolongation structures of nonlinear evolution equations. Journal of Mathematical Physics. 1975 Jan 1;16(1):1-7.

[2] Estabrook FB, Wahlquist HD. Prolongation structures of nonlinear evolution equations. II. Journal of Mathematical Physics. 1976 Jul 1;17(7):1293-7.

[3] Rogers C, Shadwick WF. Bäcklund transformations and their applications. (No Title). 1982.

[4] Sasaki R. Soliton equations and pseudospherical surfaces. Nuclear Physics B. 1979 Jul 23;154(2):343-57.

[5] Aktosun T, Demontis F, Van der Mee C. Exact solutions to the sine-Gordon equation. Journal of Mathematical Physics. 2010 Dec 1;51(12).

[6] Santos CD, Rubiera-Garcia D. Generalized sine-Gordon solitons. Journal of Physics A: Mathematical and Theoretical. 2011 Oct 21;44(42):425402.

[7] Bracken P. An exterior differential system for a generalized Korteweg-de Vries equation and its associated integrability. Acta applicandae mathematicae. 2007 Feb;95(3):223-31.

[8] Bracken P. Exterior Differential Systems for Higher Order Partial Differential Equations. Journal of Mathematics and Statistics. 2010;6(1):52-5.

[9] Bracken P. Exterior differential systems and prolongations for three important nonlinear partial differential equations. Commun. Pure Appl. Anal.. 2011 Sep 1;10:1345.

[10] Bryant RL, Chern SS, Gardner RB, Goldschmidt HL, Griffiths PA. Exterior differential systems. Springer Science & Business Media; 2013 Jun 29.

[11] Olver PJ. Applications of Lie groups to differential equations. Springer Science & Business Media; 1993.

[12] Bluman GW, Kumei S, Bluman GW. Symmetries and differential equations. New York: Springer; 1989 Aug.

[13] Anco SC, Bluman G. Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications. European Journal of Applied Mathematics. 2002 Oct;13(5):545-66.

[14] Wolf T. A comparison of four approaches to the calculation of conservation laws. European Journal of Applied Mathematics. 2002 Apr;13(2):129-52.

[15] Chern SS. Geometrical interpretation of Sinh–Gordon equation. Ann. Polon. Math. 1980;39:74.

[16] Dodd R, Fordy A. The prolongation structures of quasi-polynomial flows. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences. 1983 Feb 8;385(1789):389-429.

[17] Finley III JD, McIver JK. Infinite-dimensional Estabrook–Wahlquist prolongations for the sine-Gordon equation. Journal of Mathematical Physics. 1995 Oct 1;36(10):5707-34.

[18] Tang XY, Liang ZF. Nonlocal symmetries and conservation laws of the sinh-Gordon equation. Journal of Nonlinear Mathematical Physics. 2017 Jan 2;24(1):93-106.